Agenda for Friday, May 24, 2024

Reminders

- Office hours Fri 2-3 in Locy 203, Fri 3-4 with Kai-hsiang Locy 212
- ! MyLab on substitutions due Sun
- ! Exam 2 Tuesday

Recap

The Jacobian of a transformation T given by $\begin{cases} x = g(u,v,w) \\ y = h(u,v,w) \\ z = k(u,v,w) \end{cases}$ is the matrix determinant

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

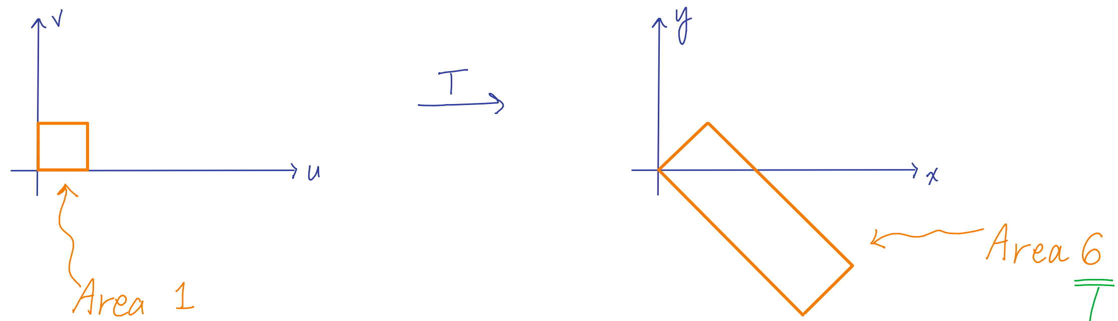
Note: For a transformation from $\mathbb{R}^2 \to \mathbb{R}^2$, use the 2×2 matrix $J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

Substitutions, cont

Geometric meaning of Jacobian

Consider the transformation $T: (u,v) \to (x,y)$ given by $x = u + 3v$, $y = u - 3v$
What happens to the unit square given by $0 \le u \le 1$, $0 \le v \le 1$ under the tranformation T?

Here's what T does to the unit square in the uv-plane:



The Jacobian of T is $J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 1 & -3 \end{vmatrix} = \underline{\underline{-6}}$

The absolute value of the Jacobian is the scale factor for the area of a region being transformed by T.

Change of variables formula for double integrals

Suppose T is a transformation that maps a region S in the uv-plane to a region R in the xy-plane. Then

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,dudv$$

Example 1. Let T be the transformation given by $x = 2u + v$, $y = u + 2v$

(a) Find the inverse transformation T^{-1}.

Solve for u and v: $x = 2u + v$, $-2(x = 2u+v)$

$-2(y = u + 2v)$, $y = u + 2v$

$x - 2y = -3v$, $-2x + y = -3u$

$v = -\frac{1}{3}(x - 2y)$, $u = \frac{1}{3}(2x - y)$

$\boxed{T^{-1} \text{ is } u = \frac{1}{3}(2x-y),\ v = -\frac{1}{3}(x-2y)}$

(b) Compute the Jacobians of T and T^{-1}.

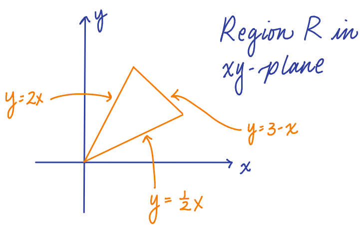
Jacobian of $T = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$

Jacobian of $T^{-1} = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{vmatrix} = \frac{4}{9} - \frac{1}{9} = \frac{1}{3}$

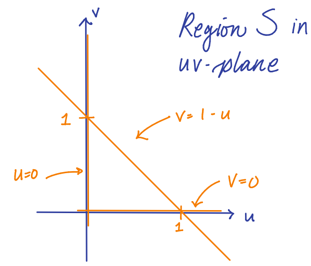
Remark. If a transformation T has Jacobian J, then the inverse transformation T^{-1} has Jacobian $\frac{1}{J}$.

Example 2. Use the change of variables $x = 2u + v$, $y = u + 2v$ to evaluate the integral $\iint_R x - 3y\, dA$ where R is the triangular region with vertices $(0, 0)$, $(2, 1)$, and $(1, 2)$.

Start with a sketch



$$T = \begin{cases} x = 2u+v \\ y = u+2v \end{cases}$$



Plug each boundary piece in to the given change of variables to find the shape of the new region S in uv-plane.

Piece 1	Piece 2	Piece 3
$y = 2x$	$y = \frac{1}{2}x$	$y = 3 - x$
$u+2v = 2(2u+v)$	$u+2v = \frac{1}{2}(2u+v)$	$u+2v = 3-(2u+v)$
$u = 0$	$v = 0$	$v = 1-u$

To use the change of variables formula, we need the Jacobian of T

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$$

Write the new integral in terms of u.v . Use sketch to set the bounds.

$$\iint_R x - 3y\, dA = \iint_S \left(2u+v - 3(u+2v)\right) \cdot |3|\, dA$$

$$= 3\iint_S -u - 5v\, dA$$

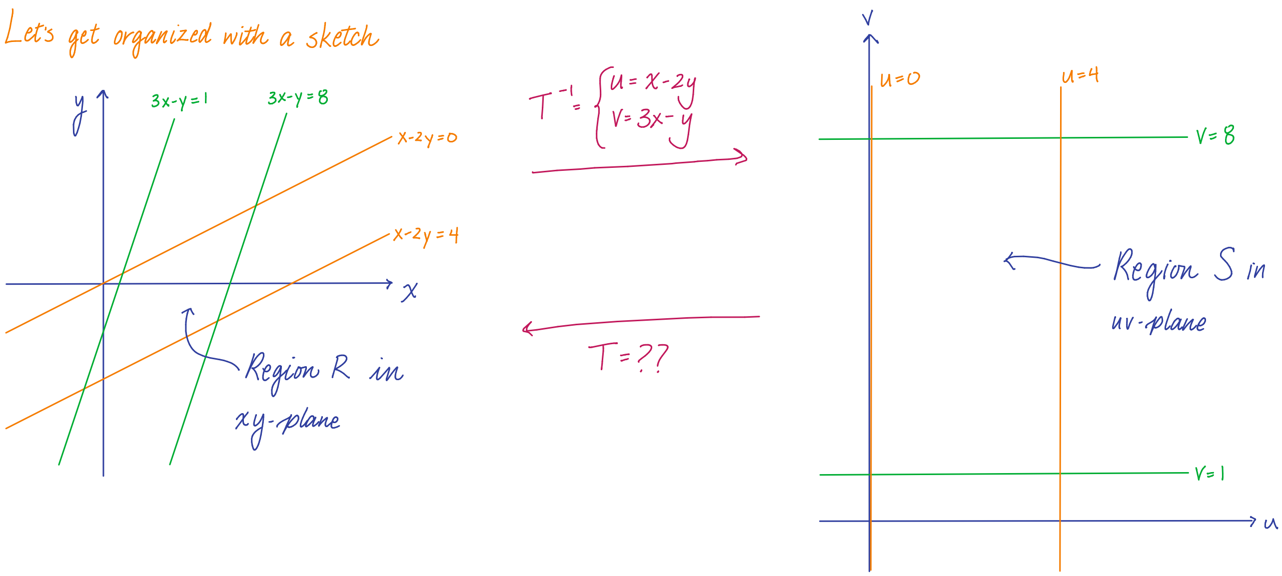
$$= 3\int_0^1\int_0^{1-u} -u-5v\, dv\, du$$

$$= \boxed{-3}$$

Example 3. Use a change of variables to compute $\displaystyle\iint_R \frac{x-2y}{3x-y}\,dA$ where R is the region bounded by $x-2y=0$, $x-2y=4$, $3x-y=1$, $3x-y=8$.

Notice $x-2y=0$ and $x-2y=4$ can be conveniently renamed $u=0$ and $u=4$ for $u=x-2y$
Similarly, $3x-y=1$ and $3x-y=8$ can be renamed $v=1$ and $v=8$ for $v=3x-y$
So let $u=x-2y$, $v=3x-y$.

Let's get organized with a sketch



We have equations for T^{-1} (from x,y to u,v) instead of T. To compute the Jacobian of T, we use the fact that the Jacobian of T is the reciprocal of the Jacobian of T^{-1}.

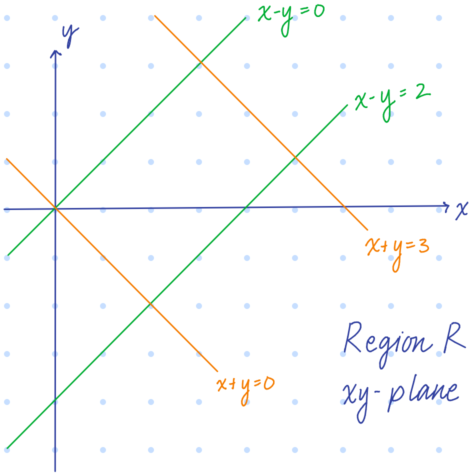
Jacobian of T^{-1} $= \left|\dfrac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} = (1)(-1)-(3)(-2) = 5$

Jacobian of T $= \frac{1}{5}$

Use this to rewrite our integral

$$\iint_R \frac{x-2y}{3x-y}\,dA = \iint_S \frac{u}{v}\cdot\left|\frac{1}{5}\right| dA = \frac{1}{5}\int_1^8\int_0^4 \frac{u}{v}\,du\,dv = \boxed{\frac{8}{5}\ln 8}$$

Evaluate $\iint_R (x+y)e^{x^2-y^2}\,dA$ where R is the rectangle enclosed by the lines $x-y=0$, $x-y=2$, $x+y=0$, $x+y=3$ by making an appropriate change of variables



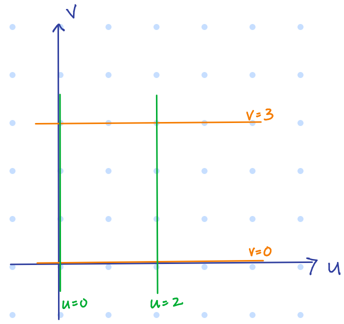
Region R in xy-plane

T??

T^{-1}

$u=x-y$

$V=x+y$



$$\text{Jacobian of } T^{-1} = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

$$\iint_R (x+y)e^{x^2-y^2}\,dA = \iint_S v e^{uv}\cdot\left|\tfrac{1}{2}\right|\,dA$$

$$= \frac{1}{2}\int_0^3\int_0^2 v e^{uv}\,du\,dv$$

$$= \frac{1}{2}\int_0^3 v\left[\frac{1}{v}e^{uv}\right]_{u=0}^{u=2} dv$$

$$= \frac{1}{2}\int_0^3 e^{2v}-1\,dv$$

$$= \frac{1}{2}\left[\frac{1}{2}e^{2v}-v\right]_0^3$$

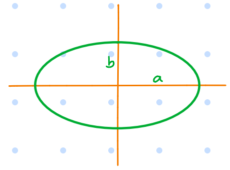
$$= \frac{1}{2}\left[\frac{1}{2}e^6-3-\frac{1}{2}+0\right]$$

$$= \boxed{\frac{e^6-7}{4}}$$

Remark

If your region of integration is an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



you can use the transformation $u = \frac{1}{a}x$, $v = \frac{1}{b}y$ to turn the region into a circle of radius 1.